

Complete integrability of geodesic motion in Sasaki-Einstein toric $Y^{p,q}$ spaces

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May 18, 2015

Abstract

We construct explicitly the constants of motion for geodesics in the 5-dimensional Sasaki-Einstein spaces $Y^{p,q}$. To carry out this task we use the knowledge of the complete set of Killing vectors and Killing-Yano tensors on these spaces. In spite of the fact that we generate a multitude of constants of motion, only five of them are functionally independent implying the complete integrability of geodesic flow on $Y^{p,q}$ spaces. In the particular case of the homogeneous Sasaki-Einstein manifold $T^{1,1}$ the integrals of motion have simpler forms and the relations between them are described in detail.

Keywords: Sasaki-Einstein spaces, Killing tensors, complete integrability.

PACS Nos: 11.30-j; 11.30.Ly; 04.50.Gh

1 Introduction

In the last years, there has been a considerable interest in finding explicit examples of Sasaki-Einstein spaces which provide supersymmetric background relevant to the AdS/CFT conjecture [1]. Among them, we mention the construction of inhomogeneous Sasaki-Einstein metrics [2, 3]. In particular, an interesting class is represented by the toric contact structures on $S^2 \times S^3$ denoted $Y^{p,q}$, where $q < p$ are positive integers.

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The purpose of this paper is the explicit construction of the constants of motion for geodesics in $Y^{p,q}$ spaces. This goal is achieved using the complete set of Killing vectors and Killing-Yano tensors on these 5-dimensional toric Sasaki-Einstein spaces. The importance of Killing forms comes from the fact that it is possible to associate with them Stäckel-Killing tensors of rank 2. In this way we have conserved quantities for geodesic motion associated with manifest and also hidden symmetries.

In spite of the fact that we are able to construct many integrals of motion, it turns out that only a part of them are functionally independent. We get that in general on $Y^{p,q}$ spaces there are only five independent constants of motion, providing the complete integrability of geodesic flow, but not superintegrability.

In order to get a better understanding of the relations between various constants of motion, we resort to the particular case of the homogeneous Sasaki-Einstein manifold $T^{1,1}$. The constants of geodesic motion on $T^{1,1}$ have simpler forms and the relations among them is explicitly described.

The paper is organized as follows. In the next Section we introduce the main concepts and technical tools we use in the rest of the paper. We give the necessary preliminaries regarding Killing tensors and toric Sasaki-Einstein manifolds. In Section 3 we investigate the constants of motion for geodesics on $Y^{p,q}$ spaces proving the complete integrability. In Section 4 we resume the construction of integrals of motion on the homogeneous Sasaki-Einstein space $T^{1,1}$. The expressions of constants of motion are simpler allowing us to write explicitly the relations among them. Finally, our conclusions are presented within the last Section.

2 Preliminaries

2.1 Stäckel-Killing and Killing-Yano tensors

Killing vector fields represent a basic object of differential geometry connected with the infinitesimal isometries. Let (M, g) be an n -dimensional Riemannian manifold with the metric g and let ∇ be its Levi-Civita connection. The vector field K_μ preserves the metric g if it satisfies Killing's equation

$$\nabla_{(\mu} K_{\nu)} = 0. \quad (1)$$

Here and elsewhere a round bracket denotes a symmetrization over the indices within.

The geodesic quadratic Hamiltonian which parametrizes the inverse metric is:

$$H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu, \quad (2)$$

where P_μ are canonical momenta conjugate to the coordinates x^μ , $P_\mu = g_{\mu\nu} \dot{x}^\nu$ with overdot denoting proper time derivative.

In the presence of a Killing vector, the system of a free particle admits a conserved quantity

$$K = K_\mu \dot{x}^\mu, \quad (3)$$

which commutes with the Hamiltonian (2) in the sense of Poisson brackets:

$$\{K, H\} = 0. \quad (4)$$

A natural generalization of Killing vector fields is represented by Stäckel-Killing tensors. A Stäckel-Killing tensor is a covariant symmetric tensor field satisfying the generalized Killing equation

$$\nabla_{(\lambda} K_{\mu_1, \dots, \mu_r)} = 0. \quad (5)$$

Stäckel-Killing tensors may be identified with first integrals of the Hamiltonian geodesic flow, which are homogeneous polynomials in the momenta. Again, using (4) and the generalized Killing equation (5) we get that the quantities

$$K_{SK} = K_{\mu_1 \dots \mu_r} \dot{x}^{\mu_1} \dots \dot{x}^{\mu_r}, \quad (6)$$

are constants of geodesic motion.

The next most simple objects that can be studied in connection with the symmetries of a manifold after the Killing vectors and Stäckel-Killing tensors are the Killing forms or Killing-Yano tensors. These are differential r -forms satisfying the equation

$$\nabla_{(\mu} f_{\nu_1) \nu_2 \dots \nu_r} = 0. \quad (7)$$

It can be easily verified that along every geodesic in M , $f_{\mu \nu_2 \dots \nu_r} \dot{x}^\mu$ is parallel.

Let us note that there is an important connection between these two generalizations of the Killing vectors. To wit, given two Killing-Yano tensors $f^{(1)}_{\nu_1 \nu_2 \dots \nu_r}$ and $f^{(2)}_{\nu_1 \nu_2 \dots \nu_r}$ there is a Stäckel-Killing tensor of rank 2

$$K_{\mu\nu}^{(1,2)} = f^{(1)}_{\mu\lambda_2 \dots \lambda_r} f^{(2)}_{\nu \lambda_2 \dots \lambda_r} + f^{(2)}_{\mu\lambda_2 \dots \lambda_r} f^{(1)}_{\nu \lambda_2 \dots \lambda_r}. \quad (8)$$

This fact offers a method to generate higher order integrals of motion by identifying the complete set of Killing-Yano tensors. We shall use this technique in the case of the toric Sasaki-Einstein $Y^{p,q}$ spaces for which the complete set of Killing-Yano tensors is known [4, 5].

The multitude of Killing-Yano tensors allows us to construct the conserved quantities for the geodesic motion and investigate the integrability of the $Y^{p,q}$ geometry.

Let us recall that in classical mechanics a Hamiltonian system with Hamiltonian H (2) and integrals of motion K_j is called *completely integrable* (or Liouville integrable) if it allows n integrals of motion H, K_1, \dots, K_{n-1} which are well-defined functions on the phase space, in involution

$$\{H, K_j\} = 0, \quad \{K_j, K_k\} = 0, \quad j, k = 1, \dots, n-1, \quad (9)$$

and functionally independent. A system is *superintegrable* if it is completely integrable and allows further functionally independent integrals of motion.

2.2 Toric Sasaki-Einstein spaces

A $(2n-1)$ -dimensional manifold M is a *contact manifold* if there exists a 1-form η (called a *contact 1-form*) on M such that:

$$\eta \wedge (d\eta)^{n-1} \neq 0. \quad (10)$$

For every choice of contact 1-form η there exists a unique vector field K_η , called the *Reeb vector field*, which satisfies:

$$\eta(K_\eta) = 1 \quad \text{and} \quad K_\eta \lrcorner d\eta = 0, \quad (11)$$

where \lrcorner is the operator dual to the wedge product.

A contact Riemannian manifold (M, g_M) is Sasakian if its metric cone $C(M)$

$$C(M) \cong \mathbb{R}_+ \times M, \quad g_{C(M)} = dr^2 + r^2 g_M, \quad (12)$$

is Kähler [6] with Kähler form [7, 8]

$$\omega = \frac{1}{2}d(r^2\eta) = r dr \wedge \eta + \frac{1}{2}r^2 d\eta. \quad (13)$$

Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line \mathbb{R}_+ . The Sasakian manifold (M, g_M) is naturally isometrically embedded into the metric cone via the inclusion

$$M = \{r = 1\} = \{1\} \times M \subset C(M). \quad (14)$$

In the case of Sasaki-Einstein manifolds there is a well known fact that the following statements are equivalent [9]:

- (i) (M, g_M) is Sasaki-Einstein with $\text{Ric } g_M = 2(n-1)g_M$;
- (ii) The Kähler cone $(C(M), g_{C(M)})$ is Ricci-flat ($\text{Ric } g_{C(M)} = 0$), i.e. a Calabi-Yau manifold;
- (iii) The transverse Kähler structure to the Reeb foliation \mathcal{F}_{K_η} is Kähler-Einstein with $\text{Ric } g^T = 2ng^T$.

The metric cone $C(M)$ is called toric if the standard n -torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ acts effectively on it, preserving the Kähler form ω .

On a $(2n-1)$ -dimensional Sasaki manifold with the contact 1-form η there are the following Killing forms [10]:

$$\Psi_k = \eta \wedge (d\eta)^k, \quad k = 0, 1, \dots, n-1. \quad (15)$$

Let us note that in the case of the Calabi-Yau cone the holonomy is $SU(n)$ and there are *two additional* parallel forms of degree n . In order to write explicitly the additional Killing forms which correspond to these parallel forms, we

shall express the volume form of the metric cone in terms of the Kähler form (13)

$$d\mathcal{V} = \frac{1}{n!} \omega^n. \quad (16)$$

Here ω^n is the wedge product of ω with itself n times. The volume of a Kähler manifold can also be written as [10, 11]

$$d\mathcal{V} = \frac{i^n}{2^n} (-1)^{n(n-1)/2} \Omega \wedge \bar{\Omega}, \quad (17)$$

where Ω is the holomorphic $(n, 0)$ volume form of $C(M)$. The additional (real) parallel forms are given by the real and respectively imaginary part of the complex volume form.

Finally, to extract the corresponding additional Killing forms of the Einstein-Sasaki spaces, we make use of the fact that for any p -form ψ on the space M we can define an associated $p+1$ -form ψ^C on the cone $C(M)$ [10]:

$$\psi^C := r^p dr \wedge \psi + \frac{r^{p+1}}{p+1} d\psi. \quad (18)$$

3 Complete integrability on $Y^{p,q}$ spaces

Let us consider the explicit local metric of the 5-dimensional $Y(p, q)$ manifold given by the square of the line element [2, 3, 12, 13]

$$ds_{ES}^2 = \frac{1-cy}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9} (d\psi - \cos \theta d\phi)^2 + w(y) \left[d\alpha + \frac{ac-2y+cy^2}{6(a-y^2)} (d\psi - \cos \theta d\phi) \right]^2,$$

where

$$w(y) = \frac{2(a-y^2)}{1-cy}, \quad q(y) = \frac{a-3y^2+2cy^3}{a-y^2}. \quad (19)$$

This metric is Einstein with $\text{Ric } g = 4g$ for all values of the constants a, c . For $c = 0$ the metric takes the local form of the standard homogeneous metric on $T^{1,1}$ [14]. Otherwise the constant c can be rescaled by a diffeomorphism and in what follows we assume $c = 1$. For

$$0 < \alpha < 1, \quad (20)$$

we can take the range of the angular coordinates (θ, ϕ, ψ) to be $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi, 0 \leq \psi \leq 2\pi$. Choosing $0 < a < 1$, the roots y_i of the cubic equation

$$a - 3y^2 + 2y^3 = 0 \quad (21)$$

are real, one negative (y_1) and two positive (y_2, y_3). If the smallest of the positive roots is y_2 , one can take the range of the coordinate y to be

$$y_1 \leq y \leq y_2. \quad (22)$$

The Reeb vector is :[14]

$$K = 3 \frac{\partial}{\partial \psi} - \frac{1}{2} \frac{\partial}{\partial \alpha}, \quad (23)$$

and the Sasakian 1-form η is [14]:

$$\eta = -2y d\alpha + \frac{1-y}{3} (d\psi - \cos \theta d\phi). \quad (24)$$

The conjugate momenta to the coordinates $(\theta, \phi, y, \alpha, \psi)$ are:

$$\begin{aligned} P_\theta &= \frac{1-y}{6} \dot{\theta}, \\ P_\phi + \cos \theta P_\psi &= \frac{1-y}{6} \sin^2 \theta \dot{\phi}, \\ P_y &= \frac{1}{6p(y)} \dot{y}, \\ P_\alpha &= w(y) \left(\dot{\alpha} + f(y) (\dot{\psi} - \cos \theta \dot{\phi}) \right), \\ P_\psi &= w(y) f(y) \dot{\alpha} + \left[\frac{q(y)}{9} + w(y) f^2(y) \right] (\dot{\psi} - \cos \theta \dot{\phi}), \end{aligned} \quad (25)$$

where

$$\begin{aligned} f(y) &= \frac{a-2y+y^2}{6(a-y^2)}, \\ p(y) &= \frac{w(y)q(y)}{6} = \frac{a-3y^2+2y^3}{3(1-y)}. \end{aligned} \quad (26)$$

Using the momenta (25), the conserved Hamiltonian (2) becomes:

$$\begin{aligned} H &= \frac{1}{2} \left\{ 6p(y) P_y^2 + \frac{6}{1-y} \left(P_\theta^2 + \frac{1}{\sin^2 \theta} (P_\phi + \cos \theta P_\psi)^2 \right) + \frac{1-y}{2(a-y^2)} P_\alpha^2 \right. \\ &\quad \left. + \frac{9(a-y^2)}{a-3y^2+2y^3} \left(P_\psi - \frac{a-2y+y^2}{6(a-y^2)} P_\alpha \right)^2 \right\} \\ &= \frac{1-y}{12} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + \frac{\dot{y}^2}{12p(y)} + \frac{q(y)}{18} (\dot{\psi} - \cos \theta \dot{\phi})^2 \\ &\quad + \frac{w(y)}{2} [\dot{\alpha} + f(y) (\dot{\psi} - \cos \theta \dot{\phi})]^2. \end{aligned} \quad (27)$$

From the isometry $SU(2) \times U(1) \times U(1)$ of the metric (19) we have that the momenta P_ϕ, P_ψ and P_α are conserved. P_ϕ is the third component of the $SU(2)$ angular momentum and P_ψ, P_α are associated to the $U(1)$ factors. In addition, the total $SU(2)$ angular momentum

$$\vec{J}^2 = P_\theta^2 + \frac{1}{\sin^2 \theta} (P_\phi + \cos \theta P_\psi)^2 + P_\psi^2 \quad (28)$$

is also conserved.

The next conserved quantities, quadratic in momenta, will be expressed in terms of Stäckel-Killing tensors as in (6). The Stäckel-Killing tensors of rank two on $Y^{p,q}$ will be constructed from Killing-Yano tensors according to (8). For this purpose we shall use the Killing-Yano tensor Ψ_1 from (15) for $k = 1$ and the additional parallel forms of degree 2, associated with the real and imaginary parts of the holomorphic $(3, 0)$ volume form Ω of the cone $C(Y^{p,q})$.

The explicit form of the Killing-Yano tensor Ψ_1 is

$$\begin{aligned}\Psi_1 = & (1-y)^2 \sin \theta d\theta \wedge d\phi \wedge d\psi - 6dy \wedge d\alpha \wedge d\psi \\ & + 6 \cos \theta d\phi \wedge dy \wedge d\alpha - 6(1-y)y \sin \theta d\theta \wedge d\phi \wedge d\alpha.\end{aligned}\quad (29)$$

Let us call Ψ the Killing form on $Y^{p,q}$ related to the complex holomorphic $(3, 0)$ form on $C(Y^{p,q})$. The real and imaginary parts of Ψ are [4, 5]:

$$\begin{aligned}\Re \Psi = & \sqrt{\frac{1-y}{p(y)}} \\ & \times \left(\cos \psi \left[d\theta \wedge dy + 6p(y) \sin \theta d\phi \wedge d\alpha + p(y) \sin \theta d\phi \wedge d\psi \right] \right. \\ & - \sin \psi \left[\sin \theta d\phi \wedge dy - 6p(y) d\theta \wedge d\alpha - p(y) d\theta \wedge d\psi \right. \\ & \left. \left. + p(y) \cos \theta d\theta \wedge d\phi \right] \right),\end{aligned}\quad (30)$$

$$\begin{aligned}\Im \Psi = & \sqrt{\frac{1-y}{p(y)}} \\ & \times \left(\sin \psi \left[d\theta \wedge dy + 6p(y) \sin \theta d\phi \wedge d\alpha + p(y) \sin \theta d\phi \wedge d\psi \right] \right. \\ & + \cos \psi \left[\sin \theta d\phi \wedge dy - 6p(y) d\theta \wedge d\alpha - p(y) d\theta \wedge d\psi \right. \\ & \left. \left. + p(y) \cos \theta d\theta \wedge d\phi \right] \right).\end{aligned}\quad (31)$$

The first Stäckel-Killing tensor $(K1)_{\mu\nu}$ is constructed according to (8) using the real part of the Killing form Ψ :

$$(K1)_{\mu\nu} = (\Re \Psi)_{\mu\lambda} (\Re \Psi)^\lambda{}_\nu \quad (32)$$

and has the non-vanishing components:

$$\begin{aligned}
K1_{\theta\theta} &= 6(1-y), \\
K1_{\phi\phi} &= \frac{3+a-6y+2y^3+(-3+a+6y-6y^2+2y^3)\cos 2\theta}{1-y}, \\
K1_{\phi\alpha} &= K1_{\alpha\phi} = -12\frac{(a+(-3+2y)y^2)\cos\theta}{1-y}, \\
K1_{\phi\psi} &= K1_{\psi\phi} = -2\frac{(a+(-3+2y)y^2)\cos\theta}{1-y}, \\
K1_{yy} &= 18\frac{1-y}{a+(-3+2y)y^2}, \\
K1_{\alpha\alpha} &= 72\frac{a+(-3+2y)y^2}{1-y}, \\
K1_{\alpha\psi} &= K1_{\psi\alpha} = 12\frac{a+(-3+2y)y^2}{1-y}, \\
K1_{\psi\psi} &= 2\frac{a+(-3+2y)y^2}{1-y}.
\end{aligned} \tag{33}$$

The corresponding conserved quantity (6) is

$$\begin{aligned}
K1 &= 6(1-y)\dot{\theta}\dot{\theta} + \frac{3+a-6y+2y^3+(-3+a+6y-6y^2+2y^3)\cos 2\theta}{1-y}\dot{\phi}\dot{\phi} \\
&\quad - 24\frac{(a+(-3+2y)y^2)\cos\theta}{1-y}\dot{\phi}\dot{\alpha} - 4\frac{(a+(-3+2y)y^2)\cos\theta}{1-y}\dot{\phi}\dot{\psi} \\
&\quad + 18\frac{1-y}{a+(-3+2y)y^2}\dot{y}\dot{y} + 72\frac{a+(-3+2y)y^2}{1-y}\dot{\alpha}\dot{\alpha} \\
&\quad + 24\frac{a+(-3+2y)y^2}{1-y}\dot{\alpha}\dot{\psi} + 2\frac{a+(-3+2y)y^2}{1-y}\dot{\psi}\dot{\psi}.
\end{aligned} \tag{34}$$

The next Stäckel-Killing tensor will be constructed from the imaginary part of Ψ :

$$(K2)_{\mu\nu} = (\Im\Psi)_{\mu\lambda}(\Im\Psi)^\lambda{}_\nu, \tag{35}$$

and we find that this tensor has the same components as $K1$ (33).

The mixed combination of $\Re\Psi$ and $\Im\Psi$ produces the Stäckel-Killing tensor

$$(K3)_{\mu\nu} = (\Re\Psi)_{\mu\lambda}(\Im\Psi)^\lambda{}_\nu + (\Im\Psi)_{\mu\lambda}(\Re\Psi)^\lambda{}_\nu, \tag{36}$$

but it proves that all components of this tensor vanish.

Finally we construct the Stäckel-Killing tensor from the Killing form Ψ_1 :

$$(K4)_{\mu\nu} = (\Psi_1)_{\mu\lambda\sigma}(\Psi_1)^{\lambda\sigma}{}_\nu. \tag{37}$$

The non-vanishing components of this tensor are:¹

$$\begin{aligned}
K4_{\theta\theta} &= 108(1-y), \\
K4_{\phi\phi} &= 18 \frac{7+a-18y+12y^2-2y^3+(1+a-6y+6y^2-2y^3)\cos 2\theta}{1-y}, \\
K4_{\phi\alpha} &= K4_{\alpha\phi} = -216 \frac{(a+(-4+5y-2y^2)y)\cos\theta}{1-y}, \\
K4_{\phi\psi} &= K4_{\psi\phi} = -36 \frac{(a-(2-y)^2(-1+2y))\cos\theta}{1-y}, \\
K4_{yy} &= 324 \frac{1-y}{a+(-3+2y)y^2}, \\
K4_{\alpha\alpha} &= 1296 \frac{a+(1-2y)y^2}{1-y}, \\
K4_{\alpha\psi} &= K4_{\psi\alpha} = 216 \frac{a+(-4+5y-2y^2)y}{1-y}, \\
K4_{\psi\psi} &= 36 \frac{a-(2-y)^2(-1+2y)}{1-y}.
\end{aligned} \tag{38}$$

and the corresponding conserved quantity is:

$$\begin{aligned}
K4 &= 108(1-y)\dot{\theta}\dot{\theta} \\
&+ 18 \frac{7+a-18y+12y^2-2y^3+(1+a-6y+6y^2-2y^3)\cos 2\theta}{1-y} \dot{\phi}\dot{\phi} \\
&- 432 \frac{(a+(-4+5y-2y^2)y)\cos\theta}{1-y} \dot{\phi}\dot{\alpha} \\
&- 72 \frac{(a-(2-y)^2(-1+2y))\cos\theta}{1-y} \dot{\phi}\dot{\psi} \\
&+ 324 \frac{1-y}{a+(-3+2y)y^2} \dot{y}\dot{y} + 1296 \frac{a+(1-2y)y^2}{1-y} \dot{\alpha}\dot{\alpha} \\
&+ 432 \frac{a+(-4+5y-2y^2)y}{1-y} \dot{\alpha}\dot{\psi} + 36 \frac{a-(2-y)^2(-1+2y)}{1-y} \dot{\psi}\dot{\psi}.
\end{aligned} \tag{39}$$

Having in mind that $K1 = K2$ and $K3$ vanishes, we shall verify if the set $H, P_\phi, P_\psi, P_\alpha, \vec{J}^2, K1, K4$ constitutes a functionally independent set of constants of motion for the geodesics of $Y^{p,q}$ constructing the Jacobian:

$$\mathcal{J} = \frac{\partial(H, P_\phi, P_\psi, P_\alpha, \vec{J}^2, K1, K4)}{\partial(\theta, \phi, y, \alpha, \psi, \dot{\theta}, \dot{\phi}, \dot{y}, \dot{\alpha}, \dot{\psi})}. \tag{40}$$

Evaluating the rank of this Jacobian we find:

$$\text{Rank } \mathcal{J} = 5, \tag{41}$$

¹In [13] the components of this tensors are correct, but the expression of the conserved quantity contains some misprints. Consequently, the evaluation of the number of functionally independent set of integrals of motion is affected.

which means that the system is completely integrable. In spite of the presence of the Stäckel-Killing tensors $K1$ and $K4$ the system is not superintegrable, $K1$ and $K4$ being a combination of the first integrals $H, P_\phi, P_\psi, P_\alpha, \vec{J}^2$.

In the next Section we shall analyze the integrability for the space $T^{1,1}$. In this case the formulas are not so intricate and the dependence of the corresponding Stäckel-Killing tensors $K1$ and $K4$ of the set $H, P_\phi, P_\psi, P_\alpha, \vec{J}^2$ will be worked out explicitly.

4 Complete integrability on $T^{1,1}$ space

The homogeneous Sasaki-Einstein metric on $S^2 \times S^3$ is usually referred to as $T^{1,1}$. The $T^{1,1}$ space was considered as the first example of toric Sasaki-Einstein/quiver duality [15].

The isometries of $T^{1,1}$ form the group $SU(2) \times SU(2) \times U(1)$ and the metric of this space may be written down explicitly by utilizing the fact that it is a $U(1)$ bundle over $S^2 \times S^2$. Let us denote by (θ_1, ϕ_1) and (θ_2, ϕ_2) the coordinates which parametrize the two sphere in a conventional way, and the angle $\psi \in [0, 4\pi)$ to parametrize the $U(1)$ fiber. Using these parametrizations the $T^{1,1}$ metric may be written as [16, 14]:

$$ds^2(T^{1,1}) = \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2. \quad (42)$$

The globally defined contact 1-form η is:

$$\eta = \frac{1}{3}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2), \quad (43)$$

and the Reeb vector field K_η has the form:

$$K_\eta = 3 \frac{\partial}{\partial \psi} \quad (44)$$

and it is easy to see that $\eta(K_\eta) = 1$.

The conjugate momenta to the coordinates $(\theta_1, \theta_2, \phi_1, \phi_2, \psi)$ are:

$$\begin{aligned} P_{\theta_1} &= \frac{1}{6} \dot{\theta}_1, \\ P_{\theta_2} &= \frac{1}{6} \dot{\theta}_2, \\ P_{\phi_1} &= \frac{1}{6} \sin^2 \theta_1 \dot{\phi}_1 + \frac{1}{9} \cos^2 \theta_1 \dot{\phi}_1 + \frac{1}{9} \cos \theta_1 \dot{\psi} + \frac{1}{9} \cos \theta_1 \cos \theta_2 \dot{\phi}_2, \\ P_{\phi_2} &= \frac{1}{6} \sin^2 \theta_2 \dot{\phi}_2 + \frac{1}{9} \cos^2 \theta_2 \dot{\phi}_2 + \frac{1}{9} \cos \theta_2 \dot{\psi} + \frac{1}{9} \cos \theta_1 \cos \theta_2 \dot{\phi}_1, \\ P_\psi &= \frac{1}{9} \dot{\psi} + \frac{1}{9} \cos \theta_1 \dot{\phi}_1 + \frac{1}{9} \cos \theta_2 \dot{\phi}_2, \end{aligned} \quad (45)$$

and the conserved Hamiltonian (2) takes the form:

$$\begin{aligned}
H &= 3 \left[P_{\theta_1}^2 + P_{\theta_2}^2 + \frac{1}{\sin^2 \theta_1} (P_{\phi_1} - \cos \theta_1 P_\psi)^2 + \frac{1}{\sin^2 \theta_2} (P_{\phi_2} - \cos \theta_2 P_\psi)^2 \right] \\
&\quad + \frac{9}{2} P_\psi^2 \\
&= \frac{1}{12} (\dot{\theta}_1^2 + \sin^2 \theta_1 \dot{\phi}_1^2 + \dot{\theta}_2^2 + \sin^2 \theta_2 \dot{\phi}_2^2) + \frac{1}{18} (\dot{\psi} + \cos \theta_1 \dot{\phi}_1 + \cos \theta_2 \dot{\phi}_2)^2.
\end{aligned} \tag{46}$$

Taking into account the isometries of $T^{1,1}$, the momenta P_{ϕ_1}, P_{ϕ_2} and P_ψ are conserved. On the other hand two total $SU(2)$ angular momenta are also conserved:

$$\begin{aligned}
\vec{J}_1^2 &= P_{\theta_1}^2 + \frac{1}{\sin^2 \theta_1} (P_{\phi_1} - \cos \theta_1 P_\psi)^2 + P_\psi^2 \\
&= \frac{1}{36} \left[\dot{\theta}_1^2 + \sin^2 \theta_1 \dot{\phi}_1^2 \right] + \frac{1}{81} \left[\dot{\psi}^2 + \cos^2 \theta_1 \dot{\phi}_1^2 + \cos^2 \theta_2 \dot{\phi}_2^2 \right. \\
&\quad \left. + 2 \cos \theta_1 \dot{\phi}_1 \dot{\psi} + 2 \cos \theta_2 \dot{\phi}_2 \dot{\psi} + 2 \cos \theta_1 \cos \theta_2 \dot{\phi}_1 \dot{\phi}_2 \right]; \\
\vec{J}_2^2 &= P_{\theta_2}^2 + \frac{1}{\sin^2 \theta_2} (P_{\phi_2} - \cos \theta_2 P_\psi)^2 + P_\psi^2 \\
&= \frac{1}{36} \left[\dot{\theta}_2^2 + \sin^2 \theta_2 \dot{\phi}_2^2 \right] + \frac{1}{81} \left[\dot{\psi}^2 + \cos^2 \theta_1 \dot{\phi}_1^2 + \cos^2 \theta_2 \dot{\phi}_2^2 \right. \\
&\quad \left. + 2 \cos \theta_1 \dot{\phi}_1 \dot{\psi} + 2 \cos \theta_2 \dot{\phi}_2 \dot{\psi} + 2 \cos \theta_1 \cos \theta_2 \dot{\phi}_1 \dot{\phi}_2 \right].
\end{aligned} \tag{47}$$

In order to construct the Stäckel-Killing tensors on $T^{1,1}$ we shall consider its metric cone $C(T^{1,1})$. The Calabi-Yau cone over $T^{1,1}$ has the complex structure of the quadric singularity $\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$ minus the isolated singular point at the origin. From the holomorphic $(3,0)$ volume form on the Calabi-Yau cone $C(T^{1,1})$ we extract the additional Killing form Ψ on the $T^{1,1}$ making use of the prescription (18). After standard calculations we get the real and imaginary parts of Ψ [17]:

$$\begin{aligned}
\Re \Psi &= \cos \psi d\theta_1 \wedge d\theta_2 + \sin \theta_2 \sin \psi d\theta_1 \wedge d\phi_2 \\
&\quad - \sin \theta_1 \sin \psi d\theta_2 \wedge d\phi_1 \\
&\quad - \sin \theta_1 \sin \theta_2 \cos \psi d\phi_1 \wedge d\phi_2,
\end{aligned} \tag{48}$$

$$\begin{aligned}
\Im \Psi &= \sin \psi d\theta_1 \wedge d\theta_2 - \sin \theta_2 \cos \psi d\theta_1 \wedge d\phi_2 \\
&\quad + \sin \theta_1 \cos \psi d\theta_2 \wedge d\phi_1 \\
&\quad - \sin \theta_1 \sin \theta_2 \sin \psi d\phi_1 \wedge d\phi_2.
\end{aligned} \tag{49}$$

Using these Killing forms, the Stäckel-Killing tensor (32) becomes in this case:

$$(K1)_{\mu\nu} = 6 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sin^2 \theta_1 & 0 & 0 \\ 0 & 0 & 0 & \sin^2 \theta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (50)$$

As in the case of $Y^{p,q}$ spaces, we observe that the Stäckel-Killing tensor $K2$ (35) has the same components as $K1$ while $K3$ (36) vanishes.

On the other hand, from the contact form η (43) we evaluate the Killing-Yano tensor Ψ_1 [17]:

$$\begin{aligned} \Psi_1 = & \frac{1}{9} (\sin \theta_1 d\psi \wedge d\theta_1 \wedge d\phi_1 + \sin \theta_2 d\psi \wedge d\theta_2 \wedge d\phi_2 \\ & - \cos \theta_1 \sin \theta_2 d\theta_2 \wedge d\phi_1 \wedge d\phi_2 \\ & + \cos \theta_2 \sin \theta_1 d\theta_1 \wedge d\phi_1 \wedge d\phi_2). \end{aligned} \quad (51)$$

The corresponding Stäckel-Killing tensor $K4$ (37) is:

$$(K4)_{\mu\nu} = \frac{4}{3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}(3 + \cos^2 \theta_1) & \frac{4}{3} \cos \theta_1 \cos \theta_2 & \frac{4}{3} \cos \theta_1 \\ 0 & 0 & \frac{4}{3} \cos \theta_1 \cos \theta_2 & \frac{1}{3}(3 + \cos^2 \theta_2) & \frac{4}{3} \cos \theta_2 \\ 0 & 0 & \frac{4}{3} \cos \theta_1 & \frac{4}{3} \cos \theta_2 & \frac{4}{3} \end{pmatrix}. \quad (52)$$

Finally we investigate the integrability of the geodesic motion on $T^{1,1}$ and for this purpose we construct the Jacobian:

$$\mathcal{J} = \frac{\partial(H, P_{\phi_1}, P_{\phi_2}, P_{\psi}, \vec{J}_1^2, \vec{J}_2^2, K1, K4)}{\partial(\theta_1, \theta_2, \phi_1, \phi_2, \psi, \dot{\theta}_1, \dot{\theta}_2, \dot{\phi}_1, \dot{\phi}_2, \dot{\psi})}. \quad (53)$$

As expected, we get that the rank of this Jacobian is 5 implying the complete integrability of the geodesic motion on $T^{1,1}$. Therefore not all aforesaid constants of motion are functionally independent. For example we can choose the subset $(H, P_{\phi_1}, P_{\phi_2}, P_{\psi}, \vec{J}_1^2)$ as functionally independent constants of motion. It is quite simple to verify that the constants of motion $\vec{J}_2^2, K1$ and $K4$ are combinations of the chosen subset of constants:

$$\frac{1}{6} K1 = 12\mathcal{H} - \frac{2}{3} (9P_{\psi})^2, \quad (54)$$

$$\frac{3}{4} K4 = 12\mathcal{H} + \frac{2}{3} (9P_{\psi})^2, \quad (55)$$

$$6\vec{J}_2^2 = 2H + 3P_{\psi}^2 - 6\vec{J}_1^2. \quad (56)$$

Therefore the study of integrability of geodesic motion in $T^{1,1}$ reconfirms the results obtained in the previous section, making them more clear and convincing.

5 Conclusions

In this paper we presented the complete set of constants of motion for geodesics in $Y^{p,q}$ spaces. In the particular case of $T^{1,1}$ space the formulas are not so intricate and the discussion of the relations between various constants of motion is greatly simplified.

The complete integrability of geodesic equations for the spaces considered is closely related to the property of the complete separation of variables in some field equations on these spaces. The functionally independent first integrals on the phase space are in involution and the system is completely integrable in the Liouville sense.

It would be interesting to extend these investigations to other higher dimensional Sasaki-Einstein spaces relevant for predictions of the AdS/CFT correspondence [18]. The integrability of the geodesic flow on Sasaki-Einstein spaces offer new perspectives in the investigation of supersymmetries and separability of Hamilton-Jacobi, Klein-Gordon and Dirac equations on these spaces.

Acknowledgments

The authors are grateful to Vladimir Slesar and Gabriel-Eduard Vilcu for their useful comments in the early stages of this work. M.V. was supported by CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0137 while the work of E.M.B. was funded by CNCS-UEFISCDI project number PN-II-ID-PCE-2011-3-0264 and by PN 09 37 01 02/2009.

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